

# DIFFEOMORPHISM GROUPS OF NON-COMPACT MANIFOLDS ENDOWED WITH THE WHITNEY $C^\infty$ -TOPOLOGY

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**ABSTRACT.** Suppose  $M$  is a non-compact connected  $n$ -manifold without boundary,  $\mathcal{D}(M)$  is the group of  $C^\infty$ -diffeomorphisms of  $M$  endowed with the Whitney  $C^\infty$ -topology and  $\mathcal{D}_0(M)$  is the identity connected component of  $\mathcal{D}(M)$ , which is an open subgroup in the group  $\mathcal{D}_c(M) \subset \mathcal{D}(M)$  of compactly supported diffeomorphisms of  $M$ . It is shown that  $\mathcal{D}_c(M)$  is homeomorphic to  $N \times \mathbb{R}^\infty$  for an  $l_2$ -manifold  $N$  whose topological type is uniquely determined by the homotopy type of  $\mathcal{D}_0(M)$ . For instance,  $\mathcal{D}_0(M)$  is homeomorphic to  $l_2 \times \mathbb{R}^\infty$  if  $n = 1, 2$  or  $n = 3$  and  $M$  is orientable and irreducible.

## 1. INTRODUCTION

This paper is a continuation of the study of topological types of diffeomorphism groups of non-compact smooth manifolds endowed with the Whitney  $C^\infty$ -topology. Suppose  $M$  is a  $\sigma$ -compact smooth  $n$ -manifold without boundary. Let  $\mathcal{D}(M)$  denote the group of diffeomorphisms of  $M$  endowed with the Whitney  $C^\infty$ -topology (= the very-strong  $C^\infty$ -topology in [15]) and  $\mathcal{D}_0(M)$  the identity connected component of  $\mathcal{D}(M)$ . The group  $\mathcal{D}(M)$  includes the normal subgroup  $\mathcal{D}_c(M)$  consisting of diffeomorphisms with compact support.

In [2, Theorem 4, Theorem 6.8] we have shown that  $\mathcal{D}_c(M)$  is an  $(l_2 \times \mathbb{R}^\infty)$ -manifold and  $\mathcal{D}_0(M)$  is an open subgroup of  $\mathcal{D}_c(M)$ . Here  $l_2$  is the separable Hilbert space and  $\mathbb{R}^\infty$  is the direct limit of the sequence  $(\mathbb{R}^n)_{n \in \omega}$ , where  $\mathbb{R}^n$  is identified with the hyperspace  $\mathbb{R}^n \times \{0\}$  in  $\mathbb{R}^{n+1}$ .

In a series of papers [4, 5, 6] T. Banakh and D. Repovš studied topological properties of direct limits in the categories of uniform spaces. These results were applied in [1] to yield a simple criterion for recognizing topological groups homeomorphic to open subspaces of  $l_2 \times \mathbb{R}^\infty$  (see Theorem 2 in Section 2).

In this paper we apply the above criterion to obtain the following important conclusion on the group  $\mathcal{D}_0(M)$ .

**Theorem 1.** *For any non-compact  $\sigma$ -compact smooth  $n$ -manifold without boundary the group  $\mathcal{D}_c(M)$  is homeomorphic to an open subspace of  $l_2 \times \mathbb{R}^\infty$ .*

In [20] K. Mine and K. Sakai obtained the triangulation theorem of open subsets of  $l_2 \times \mathbb{R}^\infty$  (see Theorem 3 in Section 2). This means that any open subset  $U$  of  $l_2 \times \mathbb{R}^\infty$  is homeomorphic to  $N \times \mathbb{R}^\infty$  for some  $l_2$ -manifold  $N$  whose topological type is uniquely determined by the homotopy type of  $U$ . Thus we obtain the following conclusion of the group  $\mathcal{D}_0(M)$ .

**Corollary 1.** *The group  $\mathcal{D}_0(M)$  is homeomorphic to  $N \times \mathbb{R}^\infty$  for some  $l_2$ -manifold  $N$  whose topological type is uniquely determined by the homotopy type of  $\mathcal{D}_0(M)$ .*

In some specific cases we can detect the homotopy type of  $\mathcal{D}_0(M)$ .

**Corollary 2.** *Let  $M$  be a non-compact connected smooth  $n$ -manifold without boundary.*

- (1) *If  $1 \leq n \leq 2$ , then  $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$ .*
- (2) *If  $n = 3$  and the manifold  $M$  is orientable and irreducible (i.e., any smooth 2-sphere in  $M$  bounds a 3-ball in  $M$ ), then  $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$ .*
- (3) *If  $M$  is the interior  $\text{Int} X = X \setminus \partial X$  of a compact connected smooth  $n$ -manifold  $X$  with boundary, then*

$$\mathcal{D}_0(M) \approx \mathcal{D}_0(X, \partial X) \times \mathbb{R}^\infty.$$

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In particular, if  $\mathcal{D}_0(X, \partial X)$  is contractible, then  $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$ .

For example, if  $M$  is the 3-dimensional Euclidean space  $\mathbb{R}^3$  or the Whitehead contractible 3-manifold [14], then  $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$ .

## 2. OPEN SUBSPACES OF LF-SPACES

In this preliminary section we recall the criterion for recognizing topological groups homeomorphic to open subsets of  $l_2 \times \mathbb{R}^\infty$ . First we recall some necessary definitions. Below a *Polish space* means a separable completely metrizable space; a *Polish group* is a topological group whose underlying topological space is Polish.

A subgroup  $H$  of a topological group  $G$  is called *locally topologically complemented* (LTC) in  $G$  if  $H$  is closed in  $G$  and the quotient map  $q : G \rightarrow G/H = \{xH : x \in G\}$  is a locally trivial bundle. This condition is equivalent to saying that  $q$  has a local section at some point of  $G/H$ . Here, a *local section* of a map  $q : X \rightarrow Y$  at a point  $y \in Y$  means a continuous map  $s : U \rightarrow X$  defined on a neighborhood  $U$  of  $y$  in  $Y$  such that  $q \circ s = \text{id}_U$ .

Following [1], we say that a topological group  $G$  carries the *strong topology* with respect to a tower of subgroups

$$G_0 \subset G_1 \subset G_2 \subset \dots$$

if  $G = \bigcup_{n \in \omega} G_n$  and for any neighborhood  $U_n$  of the neutral element  $e$  in  $G_n$ ,  $n \in \omega$ , the group product

$$\prod_{n \in \omega} U_n = \bigcup_{n \in \omega} U_0 U_1 \dots U_n$$

is a neighborhood of  $e$  in  $G$ . In this case the topology of  $G$  coincides with the topology of the direct limit  $\text{g-lim}_{n \in \omega} G_n$  of the tower  $(G_n)_{n \in \omega}$  in the category of topological groups, which means that  $G$  carries the strongest group topology such that the identity maps  $G_n \rightarrow G$ ,  $n \in \omega$ , are continuous.

The following criterion is obtained in [1, Theorem 11].

**Theorem 2** (Banakh-Mine-Repovš-Sakai-Yagasaki). *A non-metrizable topological group  $G$  is homeomorphic to an open subset of  $\mathbb{R}^\infty$  or  $l_2 \times \mathbb{R}^\infty$  if  $G$  carries the strong topology with respect to an LTC-tower of Polish ANR-groups  $(G_n)_{n \in \omega}$ .*

Open subspaces of  $l_2 \times \mathbb{R}^\infty$  were studied in [20, 21] and the following Triangulation Theorem was obtained.

**Theorem 3** (Mine-Sakai). (1) *Each open subspace  $X$  of  $l_2 \times \mathbb{R}^\infty$  is homeomorphic to the product  $K \times l_2 \times \mathbb{R}^\infty$  for a locally finite simplicial complex  $K$ .*

(2) *Two open subspaces of  $l_2 \times \mathbb{R}^\infty$  are homeomorphic if and only if they are homotopically equivalent.*

Note that the product  $N = K \times l_2$  is an  $l_2$ -manifold and its topological type is determined by its homotopy type.

## 3. DIFFEOMORPHISM GROUPS OF NON-COMPACT MANIFOLDS

Suppose  $M$  is a non-compact  $\sigma$ -compact smooth  $n$ -manifold without boundary. We can represent  $M$  as the countable union  $M = \bigcup_{i \in \omega} M_i$  of compact  $n$ -submanifold  $M_i \subset M$ ,  $i \in \omega$ , such that  $M_i \subset \text{Int} M_{i+1}$ . Let  $M_{-1} = \emptyset$  and consider the  $n$ -submanifolds  $K_i = M \setminus \text{Int} M_i$ ,  $i \in \omega$ , of  $M$  and closed subgroups  $\mathcal{D}(M; K_i) = \{h \in \mathcal{D}(M) : h|_{K_i} = \text{id}_{K_i}\}$  of the diffeomorphism group  $\mathcal{D}(M)$  endowed with the Whitney  $C^\infty$ -topology, see [15]. Thus we obtain the group  $G = \mathcal{D}_c(M)$  and the tower  $(G_i)_{i \in \omega}$  of closed subgroups  $G_i = \mathcal{D}(M; K_i)$  of  $G$ .

The small box product  $\square_{i \in \omega} G_i$  is defined by

$$\square_{i \in \omega} G_i = \{(x_i)_{i \in \omega} \in \square_{i \in \omega} G_i : \exists k \in \omega \forall i \geq k, x_i = e\}.$$

This space is endowed with the box topology generated by the base consisting of boxes  $\square_{i \in \omega} U_i$ , where  $U_i$  is an open set of  $G_i$ . The left multiplication map

$$\pi : \square_{i \in \omega} G_i \rightarrow G, \pi(x_0, \dots, x_k, e, e, \dots) = x_0 \cdot x_1 \cdots x_k$$

is continuous ([2, Lemma 2.10]).

We shall show that the tower  $(G_i)_{i \in \omega}$  has the properties listed in Lemma 1 below. Hence, Theorem 1 now follows from Theorem 2.

**Lemma 1.** (1)  $G = \bigcup_{i \in \omega} G_i$  and the group  $G$  is not metrizable.

- (2)  $G_i$  is a separable  $l_2$ -manifold for each  $i \in \omega$ .
- (3)  $G_i$  is TLC in  $G_{i+1}$  for each  $i \in \omega$ .
- (4) The left multiplication map  $\pi : \square_{i \in \omega} G_i \rightarrow G$  admits a local section at any points. Hence, the map  $\pi$  is an open map and the group  $G$  carries the strong topology with respect to the sequence  $G_i$  ( $i \in \omega$ ).
- (5)  $G = \text{g-lim}_{\rightarrow} G_i$ .
- (6) Let  $H$  and  $H_i$  ( $i \in \omega$ ) denote the identity connected components of  $G$  and  $G_i$  ( $i \in \omega$ ) respectively. Then  $H$  is an open subgroup of  $G$  and the sequence  $H_i$  ( $i \in \omega$ ) of closed subgroups of  $H$  also have the above properties (1)  $\sim$  (5).

*Proof.* The statement (1) easily follows from the definitions of  $G$  and  $G_i$  themselves. The statement (4) follows directly from [2, Proposition 5.5 (2)]. (The proof of Theorem 6.8 in [2] assures that Proposition 5.5 can be applied to this setting.) The statement (5) now follows from (4) and [1, Proposition 1].

(2) Since  $M - \text{Int } K_i = M_i$  is compact, the group  $G_i$  is an infinite-dimensional separable Fréchet manifold (cf. [10], [19]).

(3) The assertion follows directly from the bundle theorem, Theorem 4, explained below.

(6) Since  $H_i$  is an open subgroup of  $G_i$  for each  $i \in \omega$ , it suffices to show that  $H = \bigcup_{i \in \omega} H_i$ . The group  $H$  is path-connected, since it is the identity connected component of  $G$  and the latter is locally path-connected by (2) and (4). Hence any  $h \in H$  can be joined to  $\text{id}_M$  by an arc  $A$  in  $H$ . By [2, Proposition 3.3] the compact subset  $A$  lies in  $G_n$  for some  $n \in \omega$ . This means that  $h \in H_n$ .  $\square$

For an  $n$ -submanifold  $L$  of  $M$  and a subset  $K \subset L$  let  $\mathcal{E}_K(L, M)$  denote the space of  $C^\infty$ -embeddings  $f : L \rightarrow M$  with  $f|_K = \text{id}_K$  endowed with the compact-open  $C^\infty$ -topology. There is a natural restriction map

$$r : \mathcal{D}(M, K) \rightarrow \mathcal{E}_K(L, M), \quad r(h) = h|_L.$$

The following is the classical bundle theorem in codimension 0 (cf. [7], [10], [22], [24]).

**Theorem 4.** Suppose  $K, L$  are  $n$ -submanifolds of  $M$  which are closed subsets of  $M$ ,  $K \subset \text{Int } L$  and  $\text{cl}_M(L \setminus K)$  is compact. Then, for any closed subset  $C$  of  $M$  with  $C \cap L = \emptyset$ , the restriction map  $r : \mathcal{D}(M, K) \rightarrow \mathcal{E}_K(L, M)$  has a local section

$$s : \mathcal{U} \rightarrow \mathcal{D}_0(M, K \cup C) \subset \mathcal{D}(M, K)$$

at the inclusion  $i_L : L \subset M$  such that  $s(i_L) = \text{id}_M$ .

For the proof of Corollary 2 we need a preliminary. For any pairs of spaces  $(X, A)$  and  $(Y, B)$  let  $[X, A; Y, B]$  denote the set of homotopy classes of maps of pairs. Any map of pairs  $f : (Y, B) \rightarrow (Z, C)$  induces a function

$$f_\# : [X, A; Y, B] \rightarrow [X, A; Z, C], \quad f_\# : [g] \rightarrow [fg].$$

Suppose  $L$  is a compact space and  $K$  is a closed subset of  $L$ . The inclusion maps  $H_i \subset H_{i+1}$  and  $H_i \subset H$  ( $i \in \omega$ ) induce the associated functions between pointed sets:

$$\begin{array}{ccc} [L, K; H_i, \text{id}_M] & \xrightarrow{\quad\quad\quad} & [L, K; H_{i+1}, \text{id}_M] \\ & \searrow \quad \quad \swarrow & \\ & [L, K; H, \text{id}_M] & \end{array}$$

Taking the direct limit, we obtain a function between pointed sets

$$\iota : \varinjlim [L, K; H_i, \text{id}_M] \longrightarrow [L, K; H, \text{id}_M].$$

Since any compact subset of  $H$  is included in some  $H_i$ , we have the following conclusion.

**Lemma 2.** *For any pair of compact spaces  $(L, K)$  the inclusion induced function*

$$\iota : \varinjlim [L, K; H_i, \text{id}_M] \longrightarrow [L, K; H, \text{id}_M]$$

*is a bijection.*

For  $m = 0, 1, \dots, \infty$ , a map  $f : X \rightarrow Y$  between path-connected spaces is called an  $m$ -equivalence if for some base point  $x \in X$ , the induced homomorphism on the  $k$ -th homotopy group

$$f_{\#} : \pi_k(X, x) \rightarrow \pi_k(Y, f(x))$$

is an isomorphism for  $k = 0, 1, \dots, m-1$  and an epimorphism for  $k = m$ . An  $\infty$ -equivalence is called a weak equivalence. If both  $X$  and  $Y$  have the homotopy type of CW-complexes, then every weak equivalence is a homotopy equivalence. Note that the groups  $H$  and  $H_i$  ( $i \in \omega$ ) are path-connected and have the homotopy type of CW-complexes.

**Corollary 3.** *For  $m = 0, 1, \dots, \infty$ , if each inclusion  $H_i \subset H_{i+1}$  is an  $m$ -equivalence, then so is the inclusion  $H_1 \subset H$ . For example, each  $H_i$  is contractible, then so is  $H$  and hence  $H \approx l_2 \times \mathbb{R}^\infty$ .*

**Proof of Corollary 2.** We keep the notations  $M_i, K_i, H_i$  ( $i \in \omega$ ) and  $H$ .

(1), (2) Since  $M$  is connected, we may assume that for each  $i \in \omega$  (a)  $M_i$  is connected and (b) each connected component of  $K_i = M \setminus \text{Int}M_i$  is non-compact. By Corollary 3 it suffices to show that each  $H_i$  is contractible. Note that the inclusion  $H_i \subset \mathcal{D}_0(M_i, \partial M_i)$  is a homotopy equivalence.

For  $n = 1, 2$  the assertion follows from [9], [18, Section 2.7], [23], [25], etc. In the case  $n = 3$ , if  $M_i$  is a 3-ball, then  $\mathcal{D}_0(M_i, \partial M_i)$  is contractible by the Smale conjecture [12, Appendix (1)]. If  $M_i$  is not a 3-ball, then by the assumption,  $M_i$  is an orientable Haken 3-manifold with boundary [14, 26] and  $\mathcal{D}_0(M_i, \partial M_i)$  is contractible by [11], [16], [17].

(3) Take a collar  $\partial X \times [0, 1]$  of  $\partial X = \partial X \times \{0\}$  in  $X$  and let  $M_i = X \setminus (\partial X \times [0, 1/i])$  and  $K_i = M - \text{Int}M_i = \partial X \times (0, 1/i]$  ( $i \in \mathbb{N}$ ). First we shall show that the inclusion  $H_i \subset H_{i+1}$  is a homotopy equivalence. Consider the restriction map  $\pi : H_{i+1} \rightarrow \mathcal{E}_{K_{i+1}}(K_i, M)$ . Since  $\mathcal{E}_{K_{i+1}}(K_i, M)$  is the space of embeddings of the collar  $K_i$  relative to  $K_{i+1}$ , it is seen that  $\mathcal{E}_{K_{i+1}}(K_i, M)$  is contractible. Since  $\mathcal{E}_{K_{i+1}}(K_i, M)$  is path-connected, from Theorem 4 it follows that the map  $\pi$  is onto and is a principal bundle with the structure group  $H_{i+1} \cap G_i$ . Since  $\mathcal{E}_{K_{i+1}}(K_i, M)$  is contractible and paracompact, this bundle is trivial and  $H_{i+1} \approx \mathcal{E}_{K_{i+1}}(K_i, M) \times (H_{i+1} \cap G_i)$ . Since  $H_{i+1}$  is path-connected, it follows that  $H_i = H_{i+1} \cap G_i$  and the inclusion  $H_i \subset H_{i+1}$  is a homotopy equivalence.

From Corollary 3 it follows that  $H \simeq H_1 \simeq \mathcal{D}_0(M_1, \partial M_1) \approx \mathcal{D}_0(X, \partial X)$ . Since the last one is an  $l_2$ -manifold, we have  $H \approx \mathcal{D}_0(X, \partial X) \times \mathbb{R}^\infty$  by Corollary 1.  $\square$

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